Contents

1 Algebraic Preliminaries

Readers with adequate general-algebra background may skip this section. What follows is taken from [2], chapters I and III.

1.1 Categories

We recall that a morphism between two algebraic structures is a "structure-preserving" map; common examples include linear transformations between vector spaces, homeomorphisms between topological spaces, and homomorphisms between groups, rings, and modules over rings.

A ***category*** $\mathcal{A}$ consists of the following

1. A collection of objects $\mathrm{Ob}(\mathcal{A})$;

2. for any two objects $A, B \in \mathrm{Ob}(\mathcal{A})$, there exists a set $\mathrm{Hom}(A, B)$ called the set of ***morphisms*** of A into B;

3. for any three objects $A, B, C \in \mathrm{Ob}(\mathcal{A})$, there exists a law of composition

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

satisfying the following axioms:

CAT 1. Two sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(A', B')$ are disjoint unless $A = A'$ and $B = B'$, in which case they are equal.

CAT 2. For each object A of $\mathcal{A}$, there is a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ which acts as right and left identity for the elements of $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(B, A)$, respectively, for all objects $B \in \mathrm{Ob}(\mathcal{A})$.

CAT 3. The law of composition is associative (when defined), i.e., given $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, and $h \in \mathrm{Hom}(C, D)$, then

$$(h \circ g) \circ f = h \circ (g \circ f),$$

for all objects A, B, C, D of $\mathcal{A}$.

Examples of categories include

- **Grp**, the category of groups, with group homomorphisms
- $\mathcal{C}^0$, the category of open sets in $\mathbb{R}^n$, with continuous maps

The collection of all morphisms in a category $\mathcal{A}$ will be denoted by $\mathrm{Ar}(\mathcal{A})$. A morphism $f \in \mathrm{Hom}(A, B)$ is also written $f : A \to B$, or A morphism f is called

$$A \xrightarrow{f} B$$

an ***isomorphism*** if there exists a morphism $g : B \to A$ such that $g \circ f$ and $f \circ g$ are the identities in $\mathrm{Hom}(A, A)$ and $\mathrm{Hom}(B, B)$, respectively. If $A = B$, then we also say that the isomorphism is an ***automorphism***. A morphism of an object into itself is called an ***endomorphism***.

Lastly, we define universality. Let $\mathcal{A}$ be a category. An object P of $\mathcal{A}$ is called ***universally attracting*** if there exists a unique morphism of each object of $\mathcal{A}$ into P, and called ***universally repelling*** if, for ever object of $\mathcal{A}$, there exists a unique morphism of P into this object. When the context makes our meaning clear, we shall call objects P as above ***universal***. Since a universal object P admits the identity morphism into itself, it follows that if P and P' are two universal objects in $\mathcal{A}$, then there exists a unique isomorphism between them.

1.2 Inverse Limits

Let I be a set of indices. Suppose given a relation of partial ordering in I, namely for some pairs (i, j) we have a relation $i \leq j$ satisfying the following conditions:

1. For all $i, j, k \in I$, we have $i \leq i$;
2. if $i \leq j$ and $g \leq i$, then $i = j$.

We say that I is ***directed*** if, for all $i, j \in I$, there exists k such that $i \leq k$ and $j \leq k$.

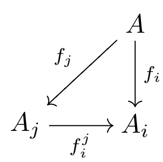
Let I be a directed system of indices. Let $\mathcal{A}$ be a category, and $\{A_i\}$ a family of objects in $\mathcal{A}$. For all $i, j \in I$ such that $j \geq i$, we suppose that we are given a morphism

$$f_i^j : A_j \to A_i$$

satisfying the relations

$$f_k^i \circ f_i^j = f_k^j \text{ and } f_i^i = \text{id},$$

whenever $j \geq i$ and $i \geq k$. An ***inverse limit*** for the family $\{f_i^k\}$ is a universal object in the following category $\mathcal{C}$: $\text{Ob}(\mathcal{C})$ consists of pairs (A, f_i) with $f_i : A \to A_i$ such that the following diagram commutes for all $i, j \in I$:



We often say that

$$A = \varprojlim A_i$$

is the inverse limit, omitting the f_j^i from the notation.

Inverse limits exist in the category of groups, in the category of modules over a ring, and in the category of rings. We refer the reader to page 162 of [2], should the reader be interested in seeing a proof.

2 Symmetric Functions

What follows is taken from [3], sections I.1 through I.3.

2.1 Partitions and Young Diagrams

A ***partition*** is a sequence of nonnegative integers in decreasing order, almost all of which are zeroes. The non-zero parts of a partition λ are called the ***parts*** of λ; two partitions are considered equal if their parts are equal. The number of parts is the ***length*** of λ, denoted by $l(\lambda)$. The sum of the parts is the ***weight*** of λ, denoted by $|\lambda|$. If $|\lambda| = n$, we say that λ is a ***partition of*** n. The set of all partitions of n is denoted by $\mathcal{P}_n$.

Let us remark on the notation. We often define a partition by specifying its terms:

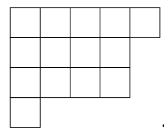
$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots);$$

it is, however, sometimes convenient to specify the number of times each integer occurs as a part:

$$\lambda = (1^{m_1} 2^{m_2} \ldots r^{m_r} \ldots),$$

indicating that i occurs m_i times in λ. The number m_i is called the ***multiplicity*** of i in λ.

The ***Young diagram*** of a partition λ is the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. We often use two-dimensional grids for diagrams; for example, the diagram of the partition (5441) is



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The ***conjugate*** of a partition λ is the partition λ' whose diagram is obtained by reflection in the main diagonal; that is, λ'_i is the number of nodes in the ith column of λ, or equivalently:

$$\lambda'_i = |\{j|\lambda_j \geq i\}|.$$

Given partitions λ and μ, we can now define the inclusion relation as follows: we say that $\lambda \supset \mu$ if the diagram of λ contains the diagram of μ. Formally speaking, $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$, considering λ and μ as infinite sequences with a sequence of zeroes tacked onto the tail-end of the sequences. The set-theoretic difference $\theta = \lambda - \mu$ is called a ***skew diagram***.

2.2 The Ring of Symmetric Functions

A polynomial $f(x_1, x_2, \ldots, x_n)$ is called a ***symmetric polynomial*** if

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

for all $\sigma \in S_n$. The collection of all symmetric polynomials of $\mathbb{Z}[x_1, \ldots, x_n]$ form a subring

$$\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n},$$

the ***ring of symmetric polynomials*** in n variables. If we define, for all nonnegative integers k, a ring Λ_n^k of the homogeneous symmetric polynomials of degree k together with the zero polynomial, then we have

$$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k,$$

which, in turn, shows that Λ_n is a graded ring; other properties are easily checked.

It is convenient to think of symmetric functions as symmetric polynomials in infinitely many variables. To make this idea precise, let us consider, for integers $m \geq n$, the homomorphism

$$\mathbb{Z}[x_1, \ldots, x_m] \to \mathbb{Z}[x_1, \ldots, x_n]$$

which sends each of $x_{n+1}, \ldots, x_m$ to zero and the other x_i to themselves. On restriction to Λ_m gives a surjective homomorphism

$$\rho_{m,n} : \Lambda_m \to \Lambda_n.$$

On restriction to Λ_m^k we have homomorphisms

$$\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$$

for all $k \geq 0$ and $m \geq n$, which are always surjective, and are bijective for $m \geq n \geq k$.

We now form the inverse limit

$$\Lambda^k = \varprojlim \Lambda_n^k$$

of the $\mathbb{Z}$-modules Λ_n^k relative to the homomorphisms $\rho_{m,n}^k$: an element of Λ^k is, by definition, a sequence $f = (f_n)_{n \geq 0}$, where each $f_n = (x_1, \ldots, x_n)$ is a homogeneous symmetric polynomial of degree k in $x_1, \ldots, x_n$, and

$$f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n)$$

whenever $m \geq n$.

We now let

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k.$$

We have surjective homomorphisms

$$\rho_n = \bigoplus_{k \geq 0} \rho_n^k : \Lambda \to \Lambda_n$$

for each $n \geq 0$, and ρ_n is an isomorphism in degrees $k \leq n$. Λ has a structure of a graded ring such that the ρ_n are ring homomorphisms, and is called the ***ring of symmetric functions*** in countably many independent variables $x_1, x_2, \ldots$; we remark that the elements of Λ are no longer polynomials, but rather formal infinite sums of monomials.

2.3 Schur Functions

Suppose to begin with that the number of variables is finite, say $x_1, \ldots, x_n$. For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Fix α, and consider the polynomial a_α obtained by antisymmetrizing x^α, viz.,

$$a_\alpha = a_\alpha(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \sigma(x^\alpha).$$

This polynomial a_α is skew-symmetric, i.e., we have

$$\sigma(a_\alpha) = \mathrm{sgn}(\sigma) a_\alpha$$

for any $\sigma \in S_n$. Consequently, a_α vanishes unless $\alpha_1, \ldots, \alpha_n$ are all distinct. Hence, we may assume that $\alpha_1 > \alpha_2 > \ldots \alpha_n \geq 0$, and therefore we may write $\alpha = \lambda + \delta$, where $\delta = (n-1, n-2, \ldots, 1, 0)$, and λ a partition of length at most n. Then we have

$$a_\alpha = a_{\lambda+\delta} = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \sigma(x^{\lambda+\delta})$$

which can be written as a determinant:

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}.$$

This determinant is divisible, in $\mathbb{Z}[x_1, \ldots, x_n]$, by each of the differences $x_i - x_j$ $(1 \leq i < j \leq n)$, and hence by their product, called the ***Vandermonde determinant***:

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta.$$

It then follows that $a_{\lambda+\delta}$ is divisible by a_δ in $\mathbb{Z}[x_1, \ldots, x_n]$, and that the quotient

$$s_\lambda = s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta$$

is ***symmetric***, i.e., is in Λ_n. The polynomial s_λ is called the ***Schur function*** in the variables $x_1, \ldots, x_n$, corresponding to the partition λ with $l(\lambda) \leq n$, and is homogeneous of degree $|\lambda|$.

Now, let us consider the effect of increasing the number of variables. If $l(\alpha) \leq n$, it is clear that $a_\alpha(x_1, \ldots, x_n, 0) = a_\alpha(x_1, \ldots, x_n)$. Hence,

$$\rho_{n+1,n}(s_\lambda(x_1, \ldots, x_{n+1})) = s_\lambda(x_1, \ldots, x_n)$$

where $\rho_{n+1,n}$ is the homomorphism defined in Section 2.2. For each partition λ, therefore, the polynomials $s_\lambda(x_1, \ldots, x_n)$ define a unique element $s_\lambda \in \Lambda$ as $n \to \infty$, which is homogeneous of degree $|\lambda|$. The resulting s_λ is the ***Schur function*** corresponding to the partition λ in the ring of symmetric functions.

3 Generalizations of Schur Functions

3.1 Preliminary Definitions

Let λ be a diagram of a partition, and $s = (i, j)$ the jth box in row i. We define the ***arm*** and the ***coarm*** of s to be the numbers of boxes to its right and left, viz.,

$$\begin{aligned} \mathrm{arm}(s) &= \lambda_i - j; \\ \mathrm{coarm}(s) &= j - 1. \end{aligned}$$

Similarly, we define the ***leg*** and the ***coleg*** of s as the numbers of boxes directly below and above s, viz.,

$$\begin{aligned} \text{leg}(s) &= \lambda'_j - i; \\ \text{coleg}(s) &= i - 1; \end{aligned}$$

where λ' is the conjugate diagram of λ.

For an indeterminate α, we now set

$$\begin{aligned} c_\lambda &= \alpha\text{arm}(s) + (\text{leg}(s) + 1); \\ c'_\lambda(s) &= \alpha(\text{arm}(s) + 1) + \text{leg}(s); \end{aligned}$$

and define the ***J-function*** j_λ by putting $j_\lambda = c_\lambda c'_\lambda$, where

$$c_\lambda = \prod_{s\in\lambda} c_\lambda(s) \text{ and } c'_\lambda = \prod_{s\in\lambda} c'_\lambda(s).$$

3.2 Interpolation Polynomials

A complex vector $\rho \in \mathbb{C}^n$ is ***dominant*** if,for all $i < j$, $\rho_i - \rho_j$ is not a negative integer. For the rest of the paper, we assume that ρ is dominant. Further, we define, for any partition λ, another partition $\bar{\lambda}$ by setting $\bar{\lambda} = \lambda + \rho$.

Given a diagram $\lambda \subset \mathbb{Z}^2$ of a partition and its conjugate λ', we define the ***ρ-hooklength*** to be

$$c^\rho_\lambda = \prod_{s\in\lambda},$$

where $c^\rho_\lambda(s) = (\lambda_i - j + 1) + (\rho_i - \rho_{\lambda'_j})$ for each $s \in \lambda$. This shall be used to specify a normalization of interpolation polynomials.

Let $\mathcal{S}(n, d)$ be the set of partitions λ with $l(\lambda) \leq n$ and $|\lambda| \leq d$. For any $\lambda \in \mathcal{S}(n, d)$ and a complex vector $\rho \in \mathbb{C}$, we define the ***interpolation polynomial*** P^ρ_λ to be the polynomial in n variables satisfying the following conditions:

IP1. P^ρ_λ is symmetric;

IP2. $\deg(P^\rho_\lambda) \leq d$;

IP3. $P^\rho_\lambda(\bar{\mu}) = 0$ for all $\mu \in \mathcal{S}(n, d)$ distinct from λ;

IP4. $P^\rho_\lambda(\bar{\lambda}) = c^\rho_\lambda$.

The existence and uniqueness of P^ρ_λ is proved in [1]; see Theorem 2.1.

Let us exhibit a simple case of interpolation polynomial. We fix $d \in \mathbb{N}$, and set $n = 1$. Then

$$\mathcal{S}(1, d) = \{(d), (d-1), (d-2), \ldots, (0)\}.$$

Pick $\lambda = (k) \in \mathcal{S}(1, d)$, and $\rho \in \mathbb{C}$. **IP1** tells us nothing, as all polynomials of one variable are symmetric. **IP2** limits the degree of the polynomial P_λ^ρ to be at most d. **IP3** specifies some roots of P_λ^ρ, viz.,

$$P_\lambda^\rho(x) = f(x) \prod_{\substack{0 \leq i \leq d \\ i \neq k}} x - (i + \rho)$$

where f is a polynomial in one variable. In particular, this implies that P_λ^ρ is at least of degree d, whence $\deg(P_\lambda^\rho) = d$. Thus, we have

$$P_\lambda^\rho(x) = C \prod_{\substack{0 \leq i \leq d \\ i \neq k}} x - (i + \rho)$$

for some $C \in \mathbb{C} \setminus \{0\}$.

It remains to determine the constant C. The diagram of λ consists of the points $(1, j)$ with $1 \leq j \leq k$, hence

$$c_\lambda^\rho = \prod_{s \in \lambda} c_\lambda^\rho(s) = \prod_{j=1}^{k} (k - j + 1) + (\rho - \rho) = k!.$$

Since we have

$$P_\lambda^\rho(\lambda + \rho) = \prod_{\substack{0 \leq i \leq d \\ i \neq k}} k - i = C \cdot k! \cdot (-1)^{d-k}(d - k)!,$$

we invoke **IP4** to conclude that

$$C = \frac{1}{(-1)^{n-k}(d - k)!}.$$

In particular, we have $C = 1$ if $k = d$.

3.3 Jack Polynomials

For the rest of the paper, we assume that $\rho = r\delta$, where $\delta = (n - 1, \ldots, 0)$ and $r \in \mathbb{C}$. We have already assumed that ρ is dominant, whence it follows that $r \neq p/q$, where p and q are positive integers such that $q < n$.

In light of Corollary 4.7 in [1], a ***Jack polynomial*** $P_\lambda^{(1/r)}$ is defined as the top homogeneous component of the interpolation polynomial $P_\lambda^{r\delta}$. The numerical parameter $1/r$ is often denoted by α; of course, $\alpha = 1/r$.

We note that $P_\lambda^{(\alpha)}$ is *not* an interpolation polynomial. If we take $\rho = 1/r$, then $n = 1$, whence P_λ^ρ is the interpolation polynomial in one variable exhibited in the previous section. This polynomial is clearly not homogeneous.

The Jack polynomial $P_\lambda^{(\alpha)}$ defined as above is always monic, and it is thus called the ***monic form***. There are few other common forms of Jack polynomials, each with different normalizations. One such form is the ***integral form*** of

Jack polynomial, denoted $J_\lambda^{(\alpha)}$; it is defined by setting

$$J_\lambda^{(\alpha)} = c_\lambda P_\lambda^{(\alpha)}.$$

Between interpolation polynomials and the integral form of Jack polynomial, the following relation holds:

$$\frac{P_\lambda^{(r\delta)}}{c_\lambda^\rho} = r^{|\lambda|} \cdot \frac{J_\lambda^{(\alpha)}}{j_\lambda} + \text{ (lower-order terms)}.$$

Since $P_\lambda^{(r\delta)} = P_\lambda^{(\alpha)}$, we have

$$P_\lambda^{(\alpha)} = \frac{J_\lambda^{(\alpha)}}{j_\lambda} \cdot r^{|\lambda|} \cdot c_\lambda^\rho,$$

whence it follows that

$$c_\lambda^{(r\delta)} = r^{|\lambda|} c'_\lambda.$$

Another normalization is the ***dual form***, defined by

$$J_\lambda^{*(\alpha)} = \frac{J_\lambda^{(\alpha)}}{j_\lambda}.$$

For the purpose of this paper, we introduce the fourth normalization $g_\lambda^{(\alpha)}$ by setting

$$g_\lambda^{(\alpha)} = \frac{P_\lambda^{(\alpha)}}{P_\lambda^{(\alpha)}(\mathbf{1})},$$

so that $g_\lambda^{(\alpha)}(\mathbf{1}) = 1$.

3.4 Bases of the Ring of Symmetric Functions

We first remark that setting $\alpha = 1$ reduces the Jack polynomial J_λ^α to a schur polynomial s_λ. Taking into consideration that all Jack polynomials are included in the corresponding interpolation polynomial, we may justifiably call interpolation polynomials generalizations of Jack polynomials, and Jack polynomials generalizations of Schur polynomials.

It is a well-known fact ([3]) that the degree k Schur polynomials in n variables form a basis for Λ_n^k. Similarly, the set $\{J_\lambda^{(\alpha)} : \lambda \in \mathcal{P}_n\}$ of Jack polynomials form a basis, called the ***Jack basis***. Of course, we can also define the ***interpolation basis***.

4 Operations on Polynomials

4.1 The Binomial Coefficients

Take the Jack basis $\{g_\lambda^{(\alpha)} : \lambda \in \mathcal{P}_n\}$. Given an indeterminate $x = (x_1, \ldots, x_n)$, we expand $g_\lambda^{(\alpha)}(x + \mathbf{1})$ in terms of the Jack basis:

$$g_\lambda^{(\alpha)}(x + \mathbf{1}) = \sum_{\mu \in \mathcal{P}_n} b_{\lambda\mu} g_\mu^{(\alpha)}(x).$$

The coefficients $b_\lambda \mu$ are called the ***binomial coefficients*** for two partitions λ and μ.

An equivalent definition is as follows:

$$b_{\lambda\mu} = \frac{P_\mu^\rho(\bar{\lambda})}{c_\mu^\rho}.$$

In other words, if we switch the normalization condition (**IP4**) of interpolation polynomials to $P_\lambda^\rho(\bar{\lambda}) = 1$, then the binomial coefficients are simply the values of the interpolation polynomial.

In some special cases, there is an explicit formula for computing $b_{\lambda\mu}$. To this end, we define the relation $:\supset$ as "differ by a single box"; in other words, if $\lambda :\supset \mu$, then $\lambda \supset \mu$, $l(\lambda) = l(\mu)$, and $|\lambda| = |\mu| + 1$. Now, if $\lambda :\supset \mu$ such that $\lambda - \mu = \{s_0\}$, we have

$$b_{\lambda\mu} = \left(\prod_{s \in C} \frac{c_\lambda(s)}{c_\mu(s)} \right) \left(\prod_{s \in R} \frac{c'_\lambda(s)}{c'_\mu(s)} \right),$$

where C and R are the other boxes in the column and row of s_0.

In all other cases, a simple recursive algorithm for computing $b_{\lambda\mu}$ can be found in [4]. We record it here as a reference:

Theorem ([4], Theorem 2)**.** *The coefficients $b_{\lambda\mu}$ satisfy the following recursions:*

1. $b_{\lambda\lambda} = 1$;
2. $(|\lambda| - |\mu|) b_{\lambda\mu} = \sum_{\nu :\supset \mu} b_{\lambda\nu} a_{\nu\mu}$;

where

$$a_{\lambda\nu} = \begin{cases} b_{\lambda\nu} & \text{if } \lambda :\supset \nu; \\ 0 & \text{otherwise.} \end{cases}$$

We define a normalization $\mathrm{Bi}(\lambda, \mu)$ of $b_{\lambda\mu}$ by

$$\mathrm{Bi}(\lambda, \mu) = b_{\lambda\mu} \cdot j_\mu.$$

It has been proven that $\mathrm{Bi}(\lambda, \mu)$ is a polynomial with integral coefficients for all partitions λ and μ. Computational results suggest the following

Conjecture. *For all partitions λ and μ, $\mathrm{Bi}(\lambda, \mu)$ has positive integral coefficients.*

4.2 The Littlewood-Richardson Coefficients

We define a polynomial h^ρ_λ by changing the normalization condition of the interpolation polynomials to $P^\rho_\lambda(\bar\lambda) = 1$, as was done in Section 4.1; in other words, we set

$$h^\rho_\lambda = \frac{P^\rho_\lambda}{P^\rho_\lambda(\bar\lambda)} = \frac{P^\rho_\lambda}{c^\rho_\lambda}.$$

Now, take the interpolation basis $\{h^\rho_\lambda : \lambda \in \mathcal{P}_n\}$, and expand the product $h^\rho_\nu h^\rho_\mu$ in terms of the basis:

$$h^\rho_\nu h^\rho_\mu = \sum_{\lambda\in\mathcal{P}_n} c_{\lambda\mu\nu} h^\rho_\lambda.$$

The coefficients $c_{\lambda\mu\nu}$ are called the ***Littlewood-Richardson coefficients of the interpolation polynomials***.

The name *Littlewood-Richardson* is derived from an analogous construction for Schur polynomials. Taking the Schur basis $\{s_\lambda : \lambda \in \mathcal{P}_n\}$, we may express the product $s_\mu s_\nu$ as a linear combination of the Schur basis:

$$s_\mu s_\nu = \sum_{\lambda\in\mathcal{P}_n} c^\lambda_{\mu\nu} s_\lambda.$$

The coefficients $c^\lambda_{\mu\nu}$ are called the ***Littlewood-Richardson coefficients of the Schur polynomials***. A combinatorial rule due to Littlewood and Richardson exists, which computes $c^\lambda_{\mu\nu}$ explicitly. For the statement of the rule, see I.9 of [3].

In some special cases, there is an explicit formula for computing $c_{\lambda\mu\nu}$. First, we quote the ***Pieri formula*** from [1]:

Theorem (Pieri Formula)**.** *Let λ, μ, and ν be partitions. We suppose that $\nu = (1^k)$, and $|\lambda| = |\mu| + |\nu|$. Then we have*

$$P^{(\alpha)}_\nu P^{(\alpha)}_\mu = \sum_\lambda \psi'_{\lambda/\mu} P^{(\alpha)}_\lambda,$$

where

$$\psi'_{\lambda/\mu}(\alpha) = \prod_{s\in\mathcal{X}(\lambda/\mu)} \frac{c_\lambda(s)/c'_\lambda(s)}{c_\mu(s)/c'_\mu(s)}$$

with $\mathcal{X}(\lambda/\mu)$ denoting the set of all boxes $(i,j) \in \lambda$ such that $\mu_i = \lambda_i$ and $\mu'_j < \lambda'_j$.

Recalling that $h^\rho_\lambda = P^\rho_\lambda / c^\rho_\lambda$, we may deduce from the definition of the Littlewood-Richardson coefficients of the interpolation polynomials that

$$\frac{P^\rho_\mu P^\rho_\nu}{c^\rho_\mu c^\rho_\nu} = \sum_\lambda c_{\lambda\mu\nu} \frac{P^\rho_\lambda}{c^\rho_\lambda}.$$

By the Pieri formula, we have

$$\frac{1}{c^\rho_\mu c^\rho_\nu} \sum_\lambda \psi'_{\lambda/\mu} P^{(\alpha)}_\lambda = \sum_\lambda \frac{c_{\lambda\mu\nu}}{c^\rho_\lambda} P^\rho_\lambda.$$

Hence, we conclude that

$$\psi'_{\lambda/\mu} = \frac{c_{\lambda\mu\nu}}{c^{\rho}_{\lambda}},$$

or

$$c_{\lambda\mu\nu} = c^{\rho}_{\lambda} \cdot \prod_{s \in \mathcal{X}(\lambda/\mu)} \frac{c_\lambda(s)/c'_\lambda(s)}{c_\mu(s)/c'_\mu(s)},$$

given that $\nu = (1^k)$, and $|\lambda| = |\mu| + |\nu|$.

In all other cases, a simple recursive algorithm for computing $c_{\lambda\mu\nu}$ can be found in [4]. See Theorem 4 in [4] for the statement of the theorem, and Section 3.2 for the proof.

We define a normalization $\mathrm{Lrc}(\lambda, \mu, \nu)$ by

$$\mathrm{Lrc}(\lambda, \mu, \nu) = c_{\lambda\mu\nu} \cdot j_\mu \cdot j_\nu.$$

A known combinatorial result is the following

Theorem. *If*

1. $\lambda \supset \mu$ *and* $\lambda \supset \nu$*;*

2. $|\lambda| \leq |\mu| + |\nu|$*,*

then $\mathrm{Lrc}(\lambda, \mu, \nu) \neq 0$*. Otherwise,* $\mathrm{Lrc}(\lambda, \mu, \nu) = 0$*.*

Noting the striking similarities between the constraints given above and those given in the Pieri formula, we may hope for the following

Conjecture. *There exists an explicit formula for* $\mathrm{Lrc}(\lambda, \mu, \nu)$ *if*

1. $\lambda \supset \mu$ *and* $\lambda \supset \nu$*;*

2. $|\lambda| \leq |\mu| + |\nu|$*.*

Since it is proven that Bi always produces polynomials with integral coefficients, one may ask whether Lrc does the same; in fact, computational results suggest the following

Conjecture. *For all partitions L, M, and N,* $\mathrm{Lrc}(L, M, N)$ *is a polynomial with positive integral coefficients.*

Unlike Bi, however, it is not even proven that Lrc is always a polynomial. Hence, it may be a reasonable first step to investigate the following

Conjecture. *For all partitions L, M, and N,* $\mathrm{Lrc}(L, M, N)$ *is a polynomial.*

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