Supplementary Note: Week 1

Real Numbers

Number Systems

We begin by establishing the different notions of numbers used in mathematics. The familiar "one, two, three, four..." form the set of *natural numbers*. Note that zero is not in the set; if we add zero, we get the set of *whole numbers*: $\{0, 1, 2, 3, 4, ...\}$.

The set of whole numbers suffices for counting ("There are **five** apples in the basket.") and ordering (" *The Goblet of Fire* is the **fourth** book in the *Harry Potter* saga."), but is nowhere near sufficient for mathematical purposes: what does $3 - 7$ equal to? We amend this issue by appending to the whole numbers a "flipped" copy of the natural numbers; namely, the negative numbers. The resulting set, $\{\ldots, -3, -2, 1, 0, 1, 2, 3, \ldots\}$, is called the set of *integers*. The set of natural numbers can then be aptly named as the set of *positive integers*. Similarly, the set of whole numbers can also be called as the set of *nonnegative integers*.

The set of integers is less likely to face troubles in arithmetic operations, as it is closed under addition, subtraction, and multiplication. This is to say that adding, subtracting, or multiplying two integers yields another integer. Even so, operations such as $3 \div 5$ produces a non-integral result: $\frac{3}{5}$, in this case. Since the concept of fractions is hardly foreign ("**half** an hour"), it makes sense to incorporate fractions into our number system. We do so by appending another copy of the integers to the set of integers; that is, we define *rational numbers* to be any number that can be written as the ratio of two integers. By definition, the set of rational numbers includes all fractions with integer denominators and numerators. Numbers such as $\frac{0.5}{2}$ are also included, for $\frac{0.5}{2} = \frac{1}{4}$.

It should be pointed out that all rational numbers can be written as *repeating decimals*. The representation can be as simple as

$$\frac{1}{3} = 0.333333\cdots = 0.\overline{3},$$

or as complicated as

$$\frac{31}{49} = 0.\overline{632653061224489795918367346938775510204081},$$

but it always is a repeating decimal. It is natural, then, to consider the decimals that do not repeat. For example,

$$0.1010010001000010000010000000\ldots$$

is not a rational number, although it seems to be a perfectly conceivable number. Clearly, we need a bigger number system.

Here is a solution: we construct a set of all numbers with a decimal representation, and call it the set of *real numbers*. This set includes familiar numbers such as $\sqrt{2}$ and π—these numbers belong to a special subset of real numbers called *irrational numbers*: that is, the real numbers that are not rational numbers.

There are even bigger number systems, with apt justifications for their existence. One such example, the set of *complex numbers*, will be covered later in the course.

Basic Set Theory

We have discussed *sets* of numbers when dealing with different number systems, but what are sets? A *set*, in mathematics, is simply a collection of objects with clear membership; that is, an object is either in the set or not in the set—"sort of in the set" is not permissible. For example, *all Rutgers University students taller than six feet* is a set, whereas *all very tall Rutgers University students* is not a set; as we do not know what "very tall" exactly means.

There are two ways of constructing a set. One way is to explicitly list all the elements of the set, and to enclose them by curly brackets: {Brower, Busch, Cooper, Neilson, Tillett} is a set (in fact, it is the set of all dining halls on the New Brunswick campus). The order in which the elements appear between the curly brackets does not matter: {Brower, Busch, Cooper, Neilson, Tillett}, {Brower, Cooper, Busch, Tillett, Neilson}, and {Neilson, Busch, Brower, Tillett, Cooper} all denote the same set. Furthermore, duplicates are meaningless in this notation; {Neilson, Neilson, Neilson, Busch, Busch, Brower, Tillett, Cooper} denotes the same set as any of the previous ones.

Another way is to specify a condition that all elements share: both *all states in the United States of America* and *all real numbers between -2 and 4* are sets (the latter is an *interval*, which is discussed in the following section). We use the following format: {dummy variable : conditions that specify the elements of the set}. For example, to write the second example in the set notation, we write $\{x : -2 < x < 4\}$ and read "all x such that x is bigger than -2 and smaller than 4"; the colon in the set notation serves as "such as." Note that the choice of x is arbitrary; $\{y : -2 < y < 4\}$, $\{\Delta : -2 < \Delta < 4\}$, and $\{☺ : -2 < ☺ < 4\}$ all denote the same set.

Just as there are addition, subtraction, multiplication, and division for numbers, there are a few operations we can apply to sets. We only cover two in this note: *union* and *intersection*. Taking the *union* of two sets produces a set with elements from either of the two sets: the union of $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Since duplicates are meaningless as remarked, any element that are in both sets should only be written down once: the union of $\{2, 4, 6\}$ and $\{1, 2, 3, 6\}$ is $\{1, 2, 3, 4, 6\}$.

Taking the *intersection* of two sets, on the other hand, produces a set with elements from both sets: the intersection of $\{2, 4, 6\}$ and $\{1, 2, 3, 6\}$ is $\{2, 6\}$. Note that taking the intersection can sometimes result in a set with no element: $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ do not share any element. The set with no element is called the *empty set* and is denoted as $\emptyset$.

As is with any other mathematical operation, there are symbols for these operations. $\cup$ is used for unions, $\cap$ for intersections. The examples in the preceding two paragraphs can then be written as follows:

$$\begin{aligned}
&\{1, 2, 3, 4\} \cup \{5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\} \\
&\{1, 2, 3, 4\} \cap \{5, 6, 7, 8\} = \emptyset \\
&\{2, 4, 6\} \cup \{1, 2, 3, 6\} = \{1, 2, 3, 4, 6\} \\
&\{2, 4, 6\} \cap \{1, 2, 3, 6\} = \{2, 6\}
\end{aligned}$$

The notion of sets is of fundamental importance in mathematics. For our purposes, a special type of sets called *intervals* will be used throughout the course.

Interval Notation

An interval, intuitively, is a "chunk" of real numbers with endpoints. For example, *all real numbers between 1 and 4* is an interval. More formally, an interval is a set of real numbers defined by an inequality. The previous example can be written as $\{x : 1 < x < 4\}$, as we discussed in the previous section. There are four types of intervals, each with its own *interval notation*:

Interval Notation	Set Notation
$[a, b]$	$\{x : a \leq x \leq b\}$
(a, b)	$\{x : a < x < b\}$
$[a, b)$	$\{x : a \leq x < b\}$
$(a, b]$	$\{x : a < x \leq b\}$

where a and b are real numbers such that $a < b$.

The intervals of the type $[a, b]$ are called *closed* intervals; similarly, the intervals of the type (a, b) are called *open* intervals. Open intervals are "open" in the sense that their endpoints are "hollow"—that is, the number at each end is *not* included. Closed intervals, similarly, are "closed" in the sense that their endpoints are "filled"—that is, the number at each end is included.

The notion of intervals can be extended to the unbounded chunks of numbers such as *all real numbers bigger than 4*. We use the positive infinity symbol ∞ and the negative infinity symbol $-\infty$ to denote the unboundedness. For example, the previous example can be written as $(4, \infty)$. Note that we always use the open-ended interval notation for ∞ or $-\infty$, as neither ∞ nor $-\infty$ is included in the interval; for "infinity" is not a real number. The set of all real numbers can be written as $(-\infty, \infty)$.

Since intervals are sets, the set operations can be applied to any two intervals. Sometimes, the operations produce intervals:

$$
\begin{aligned}
&[1, 4] \cup [4, 7] = [1, 7] \\
&(1, 2) \cup (1.5, 8] = (1, 8] \\
&[-2, 2] \cap (-1, 3) = (-1, 2] \\
&(-\infty, 2.7) \cap (-4.5, \infty) = (-4.5, 2.7)
\end{aligned}
$$

But operations such as $(3, 6) \cup (-1, 2)$ do not produce a single interval. Furthermore, taking the intersection of two intervals could result in an empty set: for example, $(1, 2) \cap (2, 3) = \emptyset$ (can you see why?).